Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 153 (2006) 2905–2907
Topology and its Applications
www.elsevier.com/locate/topol

# A note on a theorem of Talagrand

S. Moll [a,*], L.M. Sánchez Ruiz [b,1]

[a] *Escuela Técnica Superior de Ingenieros Agrónomos, Universidad Politécnica de Valencia, Camino de Vera s/n, Valencia, Spain*
[b] *Escuela Técnica Superior de Ingeniería del Diseño, Universidad Politécnica, de Valencia, Camino de Vera s/n, Valencia, Spain*

Received 20 September 2005; accepted 20 December 2005

**Abstract**

We provide an elementary argument to show that if for a hemicompact $k_{\mathbb{R}}$-space $X$ the space $C_p(X)$ contains a subset $S$ which separates the points of $X$ and is dominated by irrationals, i.e. $S$ is covered by a family $\{K_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets such that $K_\alpha \subset K_\beta$ for $\alpha \leqslant \beta$, then $C_p(X)$ is also dominated by irrationals; consequently $C_p(X)$ is $K$-analytic. This fact (which fails for non-hemicompact spaces $X$) extends an old result of Talagrand.


*MSC:* 46A50; 54D20; 54H05

*Keywords:* Compact covering; Hemicompact space; $k_{\mathbb{R}}$-space; $K$-analytic space; Locally convex space (lcs)

## 1. Introduction

By $\mathbb{N}$ we denote the space of natural numbers endowed with the discrete topology and let $\mathbb{N}^* = \mathbb{N}^{\mathbb{N}}$ be endowed with the product topology. Following Tkachuk [7] a (completely regular Hausdorff) topological space $X$ will be called *dominated by irrationals* if $X$ is covered by a family $\{K_\alpha\colon \alpha \in \mathbb{N}^*\}$ of compact sets such that $K_\alpha \subset K_\beta$ for $\alpha \leqslant \beta$. A topological space $X$ is $K$-*analytic*, see [2], or [6,8], if there is an upper semi-continuous set-valued map from $\mathbb{N}^*$ with compact values in $X$ whose union is $X$. Every $K$-analytic space is dominated by irrationals [6], see also [7], but the converse implication fails in general, see [2,3].

In 1979 Talagrand in his remarkable paper [6] proved (among other things) the following three applicable theorems.

(1) If $X$ is compact, then $C_p(X)$ is $K$-analytic iff $C_p(X)$ is dominated by irrationals.
(2) If $X$ is compact, then $C_p(X)$ is $K$-analytic iff $C_c(X)$ is weakly $K$-analytic, i.e. the weak topology of $C_c(X)$ is $K$-analytic.

* Corresponding author.
*E-mail addresses:* sanmollp@mat.upv.es (S. Moll), lmsr@mat.upv.es (L.M. Sánchez Ruiz).
1 This research was supported by the project MTM 2005-01182 of the Spanish Ministry of Education and Science, and for the second named author was also supported by the grant APLANTOP, Generalitat Valenciana (Spain) 2005. The authors thank to Prof. J. Kakol for his suggestions and discussions during his stay at the Technical University of Valencia, September 2005.

0166-8641/$ – see front matter 
doi:10.1016/j.topol.2005.12.011

(3) If $X$ is compact and $C_p(X)$ contains a $K$-analytic subspace $Y$ which separates points of $X$, then $C_p(X)$ is $K$-analytic.

Very recently Tkachuk [7, Theorem 2.9], extended (1) to any completely regular Hausdorff space $X$. In [1, Proposition 2.2] Canela extended (2) to every locally compact paracompact space $X$. In this note using an elementary argument we extend (3) by showing the following

**Theorem.** *Let $X$ be a hemicompact $k_{\mathbb{R}}$-space. If $C_p(X)$ contains a subset $S$ dominated by irrationals and $S$ separates points of $X$, then $C_p(X)$ is dominated by irrationals*; *consequently $C_p(X)$ is $K$-analytic.*

By $C_p(X)$ and $C_c(X)$ we denote the space of all continuous real-valued functions on $X$ endowed with the pointwise and compact-open topology, respectively. Recall that a topological space $X$ is called *hemicompact* if $X$ is covered by a sequence $(K_n)_n$ of compact subsets such that every compact subset of $X$ is contained in some set $K_m$. A topological space $X$ is called a *$k_{\mathbb{R}}$-space* if every real-valued map defined on $X$ which is continuous on each compact subset of $X$ is continuous.

## 2. Proof of Theorem

We shall need the following simple but applicable observation (due to Tkachuk [7, Proposition 2.1]) whose easy proof is left to the reader.

**Lemma 1.** *The domination by irrationals is inherited by closed subspaces, countable products and continuous images.*

Now we are ready to prove Theorem.

**Proof of Theorem.** Let $Y$ be the algebra generated by $S$, the constant function and the set $B = \{x \in S\colon\ |x(t)| \leqslant 1, t \in X\}$.

(1) *Assume that $X$ is compact.*

Let $\varphi : Y \times B^{\mathbb{N}} \to C_p(X)$ be a continuous map defined by $\varphi(x, (x_n)_n) = x + \sum_{n=1}^{\infty} 2^{-n} x_n$ (see Talagrand [6, proof of Theorem 3.4]). By Stone–Weierstrass theorem $Y$ is dense in $C_p(X)$. By Lemma 1, the sets $Y$ and $B$ are dominated by irrationals, so $\varphi(Y \times B^{\mathbb{N}}) = C_p(X)$ is dominated by irrationals.

(2) *Assume that $X$ is a hemicompact $k_{\mathbb{R}}$-space.*

Let $(K_n)_n$ be a fundamental (increasing) sequence of compact subsets of $X$. It is clear that for any $n \in \mathbb{N}$ the set $S_n$ of restrictions $f|K_n$, $f \in S$, belongs to $C_p(K_n)$, separates points of $K_n$, and is dominated by irrationals. By part (1) each $C_p(K_n)$ is dominated by irrationals. So $\prod_n^{\infty} C_p(K_n)$ has the same property. On the other hand, since $X$ is a hemicompact $k_{\mathbb{R}}$-space, the space $C_p(X)$ is a closed subspace of $\prod_n^{\infty} C_p(K_n)$ and Lemma 1 again applies to show that $C_p(X)$ is dominated by irrationals. We complete the proof either by using the aforementioned result of Tkachuk [7, Theorem 2.9], or by the well-known fact that angelic spaces dominated by irrationals are $K$-analytic [2, Corollary 1.1]. Note that $C_p(X)$ is already angelic for hemicompact spaces $X$, see [5, Example A and Theorem 3]. □

It is worth noting that Theorem fails if $X$ is not hemicompact. Indeed, let $X$ be a discrete space with card $X = 2^{\aleph_0}$. Then $C_p(X) = \mathbb{R}^X$ is not $K$-analytic but it contains a dense countable subset.

We note also the following

**Example 2.** There exists a hemicompact space $X$ which is not a $k_{\mathbb{R}}$-space and such that $C_p(X)$ is not $K$-analytic.

Indeed, fix $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$ and put $X = \mathbb{N} \cup \{\xi\}$, where $\mathbb{N}$ is considered with the discrete topology. Since every compact subset of $X$ is finite, the space $X$ is hemicompact. On the other hand, by Lutzer and McCoy, see [4], the space $C_p(X)$ is a Baire space, i.e. of the second Baire category. Moreover $C_p(X)$ is metrizable (since $X$ is countable) and not complete (since $X$ is not discrete). Nevertheless, $C_p(X)$ is not $K$-analytic. Indeed, otherwise $C_p(X)$ would be a separable metrizable and complete space, see [8, (21), p. 64], a contradiction.

**Proposition 3.** *Let $X$ be a hemicompact $k_{\mathbb{R}}$-space. Then the following assertions are equivalent*:

(a) $C_p(X)$ *is* $K$*-analytic.*
(b) $C_c(X)$ *with the weak topology is dominated by irrationals.*
(c) $C_c(X)$ *is weakly* $K$*-analytic.*

**Proof.** (a) $\Rightarrow$ (c) follows from the obvious fact if $C_p(X)$ is $K$-analytic then $C_c(K)$ is weakly $K$-analytic for any compact $K \subset X$. The remaining non trivial part of this proof is a direct consequence of (2) and Lemma 1. $\square$

We do not know if (b) implies (c) for any completely regular Hausdorff space $X$. In [3, Example 13 and Theorem 3] a large class of spaces $C_c(X)$ is given for which the weak dual is dominated by irrationals but is not $K$-analytic.

## References

[1] M.A. Canela, Operator and function spaces which are $K$-analytic, Portugal Math. 42 (1983) 203–218.
[2] B. Cascales, On $K$-analytic locally convex spaces, Arch. Math. 49 (1987) 232–244.
[3] J.C. Ferrando, J. Kakol, M. López Pellicer, S.A. Saxon, Tightness and distinguished Fréchet spaces, J. Math. Anal. Appl., in press.
[4] D. Lutzer, J. McCoy, Category in function spaces, Pacific J. Math. 89 (1980) 1–24.
[5] J. Orihuela, Pointwise compactness in spaces of continuous functions, J. London Math. Soc. 36 (2) (1987) 143–152.
[6] M. Talagrand, Espaces de Banach faiblement $K$-analytiques, Ann. of Math. 110 (1979) 407–438.
[7] V.V. Tkachuk, A space $C_p(X)$ is dominated by irrationals if and only if it is $K$-analytic, Acta Math. Hungar. 107 (4) (2005) 253–265.
[8] M. Valdivia, Topics in Locally Convex Spaces, North-Holland, Amsterdam, 1982.